Addendum to the paper of E. Manstavicius and M.N.N Timofeev ”A functional limit theorem related to natural divisors”

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Addendum to the paper of E. Manstavičius & M.N. Timofeev "A functional limit theorem related to natural divisors"*

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In this note, we give a simple proof of the following result which generalises, and makes slightly more precise, the main result of the paper [3] of Manstavičius & Timofeev quoted in the title. As in [3], we let $\mathbf{D} = \mathbf{D}[0,1]$ stand for the space of right-continuous functions on $[0,1]$ which have left-hand limits, endowed with the Skorokhod topology. The Borel σ-algebra of $\mathbf{D}$ is denoted by $\mathcal{D}$ and we let $\mathbf{C} = \mathbf{C}[0,1]$ be the subset of $\mathbf{D}$ comprising continuous functions. Given a non-negative multiplicative function f we put

$$F(m,v) := \sum_{d|m,\, d\leqslant v} f(d), \quad F(m) := F(m,m), \quad X_n(m,t) := F(m,n^t)/F(m).$$

The sequence of probability measures $\{\mu_n\}_{n=1}^{\infty}$ is then defined by

$$\mu_n(B) = \nu_n\{m : X_n(m,\cdot) \in B\} \qquad (B \in \mathcal{D}),$$

where ν_n is the uniform measure on the set of the first n integers. Hence, with the notation of [3], $\mu_n = \nu_n \cdot X_n^{-1}$.

Theorem. *Let f be a non-negative multiplicative function such that, for all integers h,k with $0 \leqslant h \leqslant k$, all real numbers $\alpha \in]0,1[$, and suitable $\lambda(\alpha;h,k)$ we have*

$$\lim_{n\to\infty} \sum_{n^\alpha < p \leqslant n} \frac{f(p)^h}{p\{1+f(p)\}^k} = \lambda(\alpha;h,k). \tag{1}$$

Assume furthermore that

$$\lim_{\alpha\to 0} \lambda(\alpha;1,2) = \infty. \tag{2}$$

* We include here some corrections with respect to the published version.

Then the sequence of measures $\{\mu_n\}_{n=1}^{\infty}$ *converges weakly to a limit measure* μ *defined on* $\mathcal{D}$ *such that* $\mu(\mathbf{C}) = 1$.

We also remark that the same result holds if X_n is replaced in the definition of μ_n by the slightly more canonical $X \in \mathbf{D}[0,1]$ defined by $X(m,t) := F(m, m^t)/F(m)$.

An easy example of a multiplicative function f which satisfies the assumptions of our theorem but is not covered by the theorem of Manstavičius & Timofeev is provided by choosing $f(p) = 1 + (-1)^{(p-1)/2}$ for odd p.

In the case $f = \mathbf{1}$, it is clear that X_n does not converge to the Wiener process. Indeed, it follows from [2] that

$$\int_{\mathbf{D}} \varphi(t)\,\mathrm{d}\mu(\varphi) = (2/\pi)\arcsin\sqrt{t} \qquad (0 \leqslant t \leqslant 1).$$

To give a full description of the continuous measure μ in this basic case is an interesting open problem. From [5], we have that

$$\mu\{\varphi \in \mathbf{D} : \varphi(t+\alpha) > \varphi(t)\} \ll \alpha^{\delta} \qquad (0 \leqslant \alpha \leqslant 1)$$

with $\delta := 1 - (1 + \log\log 2)/\log 2 > 0$, so that in particular

$$(\forall t \in [0,1]) \quad \varphi'(t) = 0 \quad \mu\text{-a.e.}$$

Another special feature of the limiting measure μ is that, again for $f = \mathbf{1}$, the distribution function

$$w \mapsto \mu\{\varphi \in \mathbf{D} : \varphi(t) - \varphi(s) \leqslant w\}$$

is a step-function all of whose points of increase are dyadic rational numbers. This is an immediate consequence of the results of [6].

Let us now embark on the proof. By Theorem 15.1 of Billingsley [1], we know that the required result is implied by the following properties : (a) all marginal laws of finite order converge ; (b) the sequence $\{\mu_n\}_{n=1}^{\infty}$ is tight and any weak limit has support included in $\mathbf{C}$. The simplification arises from the fact that, by Theorem 15.5 of [1], assertion (b) follows from

$$(3) \qquad (\forall \varepsilon > 0) \qquad \lim_{\alpha \to 0} \limsup_{n \to \infty} \mu_n\{\varphi \in \mathbf{D}[0,1] : \omega_{\varphi}(\alpha) > \varepsilon\} = 0,$$

where $\omega_{\varphi}(\alpha) := \sup_{|s-t| \leqslant \alpha} |\varphi(s) - \varphi(t)|$ is the modulus of continuity of φ. In the case $\varphi(t) = X_n(m,t)$, we have $\omega_{\varphi}(\alpha) = Q_n(m,\alpha) := \sup_{0 \leqslant t \leqslant 1}\{X_n(t+\alpha) - X_n(t)\}$ by monotonicity, so (3) is implied by

$$(4) \qquad \lim_{n \to \infty} \nu_n\{m : Q_n(m,\alpha) > \varepsilon\} = 0.$$

However, $Q_n(m,\alpha)$ is the value at α of the concentration function of the random variable taking the values $(\log d)/\log n$ for $d|m$ with probabilities $f(d)/F(m)$. By a well-known inequality between concentration and characteristic function (see e.g. [7], lemma III.2.6.1), we infer that

$$Q(m,\alpha) \leqslant 3\alpha \log n \int_0^{1/(\alpha \log n)} |g(m,\vartheta)| \,\mathrm{d}\vartheta,$$

where we have set

$$g(m,\vartheta) := \frac{1}{F(m)} \sum_{d|m} f(d) d^{i\vartheta}.$$

In particular, since $|g(m,\vartheta)| \leqslant 1$ for all m and ϑ, we see that (4) will follow from the average-type result

$$(5)\quad \lim_{\alpha\to 0} \limsup_{n\to\infty} I_n(\alpha) = 0, \quad \text{with} \quad I_n(\alpha) := \frac{\alpha \log n}{n} \int_{1/\log n}^{1/(\alpha \log n)} \sum_{m\leqslant n} |g(m,\vartheta)| \,\mathrm{d}\vartheta.$$

We shall derive (5) as a fairly straightforward consequence of (2). Indeed, since $g(m,\vartheta)$ is for each fixed ϑ a multiplicative function of m, the Hall–Halberstam–Richert inequality (see lemma 3 of [3]) readily yields

$$\begin{aligned}
\frac{1}{n} \sum_{m\leqslant n} |g(m,\vartheta)| &\ll \exp\Big\{ \sum_{p\leqslant n} \frac{|g(p,\vartheta)| - 1}{p} \Big\} \\
&\leqslant \exp\Big\{ -\sum_{n^\alpha < p \leqslant n} \frac{f(p)\big(1-\cos(\vartheta \log p)\big)}{p\{1+f(p)\}^2} \Big\},
\end{aligned}$$

applying the inequality

$$|1 + re^{i\varphi}|/(1+r) \leqslant 1 - r(1-\cos\varphi)/(1+r)^2 \qquad (0 \leqslant r \leqslant 1,\ \varphi \in \mathbb{R})$$

with $r = f(p)$ and $\varphi = \vartheta \log p$. Put $\varepsilon_p := f(p)/\{1+f(p)\}^2$ and

$$T_n(\alpha) := \sum_{n^\alpha < p \leqslant n} \frac{\varepsilon_p}{p}, \qquad H_n(u) := \frac{1}{T_n(\alpha)} \sum_{n^\alpha < p \leqslant n^u} \frac{\varepsilon_p}{p}$$

so that, by our assumption (2), $T_n(\alpha)$ is large for small α and $n \geqslant n_0(\alpha)$, and $H_n(u)$ is a distribution function with support included in $[\alpha, 1]$. We deduce from the above that

$$\begin{aligned}
I_n(\alpha) &= \frac{1}{n} \int_\alpha^1 \sum_{m\leqslant n} |g(m, v/\alpha \log n)| \,\mathrm{d}v \\
&\ll \int_\alpha^1 \exp\Big\{ -T_n(\alpha) \int_\alpha^1 \{1 - \cos(uv/\alpha)\} \,\mathrm{d}H_n(u) \Big\} \,\mathrm{d}v \\
&\ll \int_\alpha^1 \int_\alpha^1 e^{-T_n(\alpha)\{1-\cos(uv/\alpha)\}} \,\mathrm{d}v \,\mathrm{d}H_n(u),
\end{aligned}$$

by Jensen's inequality and inversion of the order of integrations. The inner v-integral may be estimated making the change of variables $w = uv/\alpha$ and using the inequality $1 - \cos w \geqslant \|w/2\pi\|^2$ where $\|z\| := \min_{k\in\mathbb{Z}} |z-k|$. We find that it is $\ll 1/\sqrt{T_n(\alpha)}$ uniformly in $\alpha \leqslant u \leqslant 1$, whence

$$I_n(\alpha) \ll 1/\sqrt{T_n(\alpha)}. \tag{6}$$

This plainly implies (5) and thus establishes assertion (b) above.

To complete the proof of our theorem, it remains to show assertion (a). Due to the weakening of the assumptions on f, this is slightly more difficult than in [3], although the basic argument still remains fairly close to that given in [6] for the case $f = \mathbf{1}$ and in dimension 1.

Since $X_n(m,\cdot)$ has values in $[0,1]$, the convergence of the marginal laws of finite order is equivalent to the convergence of all moments

$$\mathbb{E}_n\Big(\prod_{j=1}^r X_n(\cdot,t_j)^{\nu_j}\Big) \qquad (r \geqslant 1,\ \nu_1 \geqslant 1,\ \ldots, \nu_r \geqslant 0),$$

i.e. to the fact that the average

$$A_n(t_1,\ldots,t_k) := \frac{1}{n}\sum_{m\leqslant n}\prod_{j=1}^k X_n(m,t_j) \tag{7}$$

tends to a limit for any integer k and any $t_j \in [0,1]$ $(1 \leqslant j \leqslant k)$. Furthermore, it is an immediate consequence of the Hall–Halberstam–Richert inequality and (2) that the multiplicative function $1/F(m)$ has zero mean value, hence the expression (7) has limit zero if $t_j = 0$ for some j. Since we also have, trivially, $X_n(m,1) = 1$ $(m \leqslant n)$, we may assume henceforth that $(t_1,\ldots,t_k)$ is a fixed k-tuple in $]0,1[^k$.

Let $\varepsilon \in]0,1[$ be given. We decompose each integer $m \leqslant n$ in the form $m = m_\varepsilon m'_\varepsilon$ where m_ε is the largest divisor of m with no prime factor exceeding n^{ε^2}. Then we have (see e.g. [7], exercise III.5.6)

$$\nu_n\{m : m_\varepsilon > n^\varepsilon\} \ll e^{-1/2\varepsilon} \qquad (n \geqslant 2).$$

Moreover, it has been shown in [5] (see also [4], theorems 21 and 22) that, for all $t \in]0,1[$,

$$\nu_n\{m : \exists d|m,\ n^{t-\varepsilon} < d \leqslant n^t\} \ll_t \varepsilon^\delta, \tag{8}$$

with $\delta = 1 - (1+\log\log 2)/\log 2 \approx 0{\cdot}086071 > 0$. If $m_\varepsilon \leqslant n^\varepsilon$ and m has no divisor in $]n^{t-\varepsilon}, n^t]$ then

$$\begin{aligned} F(m)X_n(m,t) &= \sum_{d|m_\varepsilon}\ \sum_{\substack{\ell|m'_\varepsilon\\ d\leqslant n^t/\ell}} f(d)f(\ell)\\ &= \sum_{d|m_\varepsilon} f(d) \sum_{\substack{\ell|m'_\varepsilon\\ d\leqslant n^t}} f(\ell) = F(m_\varepsilon)F(m'_\varepsilon)X_n(m'_\varepsilon,t), \end{aligned}$$

so $X_n(m,t) = X_n(m'_\varepsilon, t)$.

Set

$$(9) \qquad A_n(t_1,\ldots,t_k;\varepsilon) := \frac{1}{n}\sum_{m\leqslant n}\prod_{j=1}^k X_n(m'_\varepsilon, t_j).$$

It follows from the above analysis that for any fixed $(t_1,\ldots,t_k)$ under consideration we have

$$(10) \qquad A_n(t_1,\ldots,t_k) = A_n(t_1,\ldots,t_k;\varepsilon) + O(\varepsilon^\delta).$$

Arguing as in [6] and [3], we readily obtain

$$(11) \qquad A_n(t_1,\ldots,t_k;\varepsilon) = \sum_{b\in\mathcal{B}(n,\varepsilon)}\frac{1}{b}\prod_{j=1}^k X_n(b,t_j)\varrho\Big(\frac{\log(n/b)}{\varepsilon^2\log n}\Big) + o(1),$$

where $\mathcal{B}(n,\varepsilon)$ is the set of integers b all of whose prime factors lie in the interval $]n^{\varepsilon^2}, n]$ and ϱ is Dickman's function. Note that $\varrho(u) = 0$ for $u < 0$, so we need not impose any size condition upon the b.

In order to estimate the above sum, we approximate the summands by integrals over multiplicative functions of b. Put

$$\gamma_n(b,s) := \frac{1}{F(b)}\sum_{d|b} f(d)d^{-s/\log n},$$

$$\xi_n(b,t;y) := \frac{1}{2\pi i}\int_{1-iy}^{1+iy}\gamma_n(b,s)e^{ts}\frac{\mathrm{d}s}{s} \qquad (y>0).$$

By the effective Perron formula (see e.g. [7], Theorem III.2.2), we have

$$X_n(b,t) = \xi_n(b,t;y) + O\Big(\frac{1}{F(b)}\sum_{d|b}\frac{f(d)}{1+y|t-(\log d)/\log n|}\Big),$$

so that, in particular, we have $\xi_n(b,t;y) \ll 1$ in all cases and

$$X_n(b,t) = \xi_n(b,t;y) + O(\varepsilon)$$

if $y \geqslant 1/\varepsilon^2$ and b has no divisor in the range $]n^{t-\varepsilon}, n^{t+\varepsilon}]$. Inserting into (11) and taking (8) into account, we derive that, for $y \geqslant 1/\varepsilon^2$,

$$(12) \qquad A_n(t_1,\ldots,t_k;\varepsilon) = \sum_{b\in\mathcal{B}(n,\varepsilon)}\frac{1}{b}\prod_{j=1}^k \xi_n(b,t_j;y)\varrho\Big(\frac{\log(n/b)}{\varepsilon^2\log n}\Big) + O(\varepsilon^\delta) + o(1).$$

Next, we write $\varrho(u)$ as an inverse Laplace integral, introducing (see [7], theorem III.7 and lemma III.7.1)

$$\widehat{\varrho}(z) := \int_{-\infty}^{\infty} \varrho(u)e^{-uz}\,\mathrm{d}u = \frac{1}{z}\exp\Big\{\int_0^\infty \frac{e^{-z-v}}{z+v}\,\mathrm{d}v\Big\} \qquad (z \in \mathbb{C} \smallsetminus \mathbb{R}^-). \tag{13}$$

We have—see e.g. [7], equation (III.5.45)—

$$\varrho(u) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \widehat{\varrho}(z)e^{uz}\,\mathrm{d}z \qquad (u \neq 0)$$

from which we obtain that

$$\varrho(u) = \frac{1}{2\pi i}\int_{1-iy}^{1+iy} \widehat{\varrho}(z)e^{uz}\,\mathrm{d}z + O\Big(\frac{\mathrm{e}^u}{1+|u|y}\Big) \qquad (u \in \mathbb{R})^{(1)}, \tag{14}$$

since $\widehat{\varrho}(z) = 1/z + O(1/z^2)$ for $\Re e\, z = 1$, as may be seen from (13). Choosing now $y = \exp\{1/\varepsilon^3\}$, we insert this into (12), and bound the contribution of the remainder observing that, putting $\sigma := \exp\big\{-1/(2\varepsilon^3)\big\}$

$$\sum_{\substack{b\in\mathcal{B}(n,\varepsilon)\\ n^{1-\sigma}<b\leqslant n^{1+\sigma}}} \frac{1}{b} \leqslant \sum_{\substack{b_1\in\mathcal{B}(n,\varepsilon)\\ b_1\leqslant n^{1+\sigma-\varepsilon^2}}} \frac{1}{b_1} \sum_{n^{1-\sigma}/b_1<p\leqslant n^{1+\sigma}/b_1} \frac{1}{p} \ll \sqrt{\sigma}.$$

It follows that

$$\begin{aligned} &A_n(t_1,\dots,t_k;\varepsilon)\\ &\quad = \frac{1}{(2\pi i)^{k+1}}\int_{[1-iy,1+iy]^{k+1}} \widehat{\varrho}(z)e^{z/\varepsilon^2}\,\mathrm{d}z \prod_{j=1}^k e^{t_js_j}\frac{\mathrm{d}s_j}{s_j}\sum_{b\in\mathcal{B}(n,\varepsilon)} \frac{\prod_{j=1}^k \gamma_n(b,s_j)}{b^{1+z/\varepsilon^2\log n}}\\ &\quad\quad + O(\varepsilon^\delta) + o(1).\end{aligned}$$

The inner b-sum equals

$$\prod_{n^{\varepsilon^2}<p\leqslant n}\sum_{r=0}^\infty \frac{\prod_{j=1}^k \gamma_n(p^r,s_j)}{p^{r(1+z/\varepsilon^2\log n)}} = \exp\Bigg\{\sum_{n^{\varepsilon^2}<p\leqslant n} \frac{\prod_{j=1}^k\{1+f(p)p^{-s_j/\log n}\}}{\{1+f(p)\}^k p^{1+z/\varepsilon^2\log n}} + o(1)\Bigg\}$$

and, using (1), it is easy to check by partial integration that this tends to a limit as $n \to \infty$. By the theorem of dominated convergence, we obtain, with suitable $A(t_1,\dots,t_k;\varepsilon)$,

$$A_n(t_1,\dots,t_k;\varepsilon) = A(t_1,\dots,t_k;\varepsilon) + O(\varepsilon^\delta) + o(1) \qquad (n \to \infty).$$

Inserting back into (10) and letting successively $n \to \infty$ and $\varepsilon \to 0$ we obtain that $A_n(t_1,\dots,t_k)$ tends to a limit as $n \to \infty$. This establishes assertion (a) and therefore completes the proof of the theorem.

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1. The factor e^u was erroneously omitted in the printed version. This is taken care of by introducing the quantity σ below.

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